THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2020B Advanced Calculus II
Suggested Solutions for Homework 10
Date: 15 April, 2025

1. Use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F}$ across the surface S in the direction of the outward unit normal $\mathbf{n}$.

   Solution. We calculate the curl

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = (5, 2, 3)$$

   and

$$\begin{aligned}
\mathbf{r}_r &= (\cos\theta, \sin\theta, -2r) \\
\mathbf{r}_\theta &= (-r\sin\theta, r\cos\theta, 0) \\
\mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (2r^2\cos\theta, 2r^2\sin\theta, r).
\end{aligned}$$

   Hence, the surface integral is

$$\begin{aligned}
\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma &= \int_0^{2\pi} \int_0^2 (5, 2, 3) \cdot (2r^2\cos\theta, 2r^2\sin\theta, r) dr d\theta \\
&= \int_0^{2\pi} (10r^2\cos\theta + 4r^2\sin\theta + 3r) dr d\theta \\
&= \int_0^{2\pi} \left(\frac{10}{3} r^3 \cos\theta + \frac{4}{3} r^3 \sin\theta + \frac{3}{2} r^2 \right) \Big|_{r=0}^{r=2} d\theta \\
&= \int_0^{2\pi} \left(\frac{40}{3} \cos\theta + \frac{16}{3} \sin\theta + 6 \right) d\theta \\
&= \left(\frac{40}{3} \sin\theta - \frac{16}{3} \cos\theta + 6\theta \right) \Big|_{\theta=0}^{\theta=2\pi} \\
&= 12\pi.
\end{aligned}$$

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2. Use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction.

$$\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$$

   C: The ellipse $4x^2 + y^2 = 4$ in the xy-plane, counterclockwise when viewed from above.

Solution. By Stokes' Theorem, the circulation of $\mathbf{F}$ around the curve C is given by the surface integral of the flux of the curl of $\mathbf{F}$ across the surface S where S is the surface enclosed by C. That is,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma$$

where $\mathbf{n}$ is the normal pointing up in the z direction, since the curve is in counterclockwise orientation. We compute the curl

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix} = (0, 0, 2).$$

The surface S is parametrized by

$$\mathbf{r}(r, \theta) = (r\cos\theta, 2r\sin\theta, 0), \quad 0 \le r \le 1, 0 \le \theta \le 2\pi$$

where we observe that when $r = 1$, the x and y coordinates satisfy the equation defining C. Then

$$\begin{aligned} \mathbf{r}_r &= (\cos\theta, 2\sin\theta, 0) \\ \mathbf{r}_\theta &= (-r\sin\theta, 2r\cos\theta, 0) \\ \mathbf{r}_r \times \mathbf{r}_\theta &= (0, 0, 2r). \end{aligned}$$

Hence, we have

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma \\ &= \int_0^{2\pi} \int_0^1 (0, 0, 2) \cdot (0, 0, 2r) dr d\theta \\ &= \int_0^{2\pi} \int_0^1 4r dr d\theta \\ &= \int_0^{2\pi} 2 d\theta \\ &= 4\pi. \end{aligned}$$

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3. Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$, oriented counterclockwise when viewed from above. Show that

$$\oint_C 2y dx + 3z dy - x dz$$

depends only on the area of the region enclosed by C and not on the position or shape of C.

Solution. Let $\mathbf{F} = 2y\mathbf{i} + 3z\mathbf{j} - x\mathbf{k}$. Then by Stokes' Theorem,

$$\oint_C 2ydx + 3zdy - xdz = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma$$

where $\mathbf{n} = (2, 2, 1)$ is the normal vector to the plane and S is the region in the interior of C. Computing the flux, we have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3z & -x \end{vmatrix} = (3, 1, -2).$$

Evaluting the surface integral, we see

$$\begin{aligned} \oint_C 2ydx + 3zdy - xdz &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma \\ &= \iint_S (3, 1, -2) \cdot (2, 2, 1) d\sigma \\ &= \iint_S 6 d\sigma \\ &= 6\text{Area}(S) \end{aligned}$$

as required. ◀

4. Suppose $\mathbf{F} = \nabla \times \mathbf{A}$, where

$$\mathbf{A} = (y + 2z^2)\mathbf{i} + e^{xyz}\mathbf{j} + \cos(xz)\mathbf{k}.$$

Determine the flux of $\mathbf{F}$ outward through the hemisphere

$$x^2 + y^2 + z^z = 1, \quad z \geq 0.$$

Solution. By Stokes' Theorem, the flux of $\mathbf{F}$ outward through the hemisphere is given by the circulation of $\mathbf{A}$ around the circle C: $x^2 + y^2 = 1$ (taking $z = 0$ in the equation of S). C is parametrized by

$$\mathbf{r}(t) = (\cos t, \sin t, 0), \quad 0 \leq t \leq 2\pi$$

and

$$\begin{aligned} &\mathbf{r}'(t) = (-\sin t, \cos t, 0) \\ &\mathbf{A}(\mathbf{r}(t)) = (\sin t + 2 \cdot 0, e^0, \cos(\cos t \cdot 0)) = (\sin t, 1, 1). \end{aligned}$$

Hence, we have

$$\begin{aligned}\iint_{x^2+y^2+z^2=1} \mathbf{F}\cdot\mathbf{n}d\sigma &= \oint_C \mathbf{A}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt \\ &= \int_0^{2\pi} (\sin t, 1, 1)\cdot(-\sin t, \cos t, 0)dt \\ &= \int_0^{2\pi} (-\sin^2 t + \cos t)dt \\ &= \int_0^{2\pi} -\left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right) + \cos(t)dt \\ &= \left(-\frac{1}{2}t - \frac{1}{4}\sin(2t) - \sin(t)\right)\Big|_{t=0}^{t=2\pi} \\ &= -\pi.\end{aligned}$$

Note: many students used the divergence theorem for this question. They calculated that the divergence of **F** is 0 and hence concluded that the flux of **F** over the surface is 0. However, this shows that the total flux across the hemisphere (outward) and the disk on the bottom (downward) is 0, but the question is asking for the flux across the hemisphere (outward). For the flux across only the hemisphere, use Stokes' theorem as above to find that the flux is $-\pi$. ◀

5. Calculate the net outward flux of the vector field

$$\mathbf{F} = xy\mathbf{i} + (\sin xz + y^2)\mathbf{j} + (e^{xy^2} + x)\mathbf{k}$$

over the surface S surrounding the region D bounded by the planes

$$y = 0, z = 0, z = 2 - y$$

and the parabolic cylinder $z = 1 - x^2$.

Solution. By the divergence theorem, the flux is given by the value of the triple integral of $\nabla\cdot\mathbf{F}$ over D. We have

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(\sin xz + y^2) + \frac{\partial}{\partial z}(e^{xy^z} + x) = 3y$$

and so we have

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma &= \iiint_D \nabla \cdot \mathbf{F} dV \\
&= \int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} 3y dy dz dx \\
&= \int_{-1}^{1} \int_{0}^{1-x^2} \frac{3}{2} y^2 \Big|_{y=0}^{y=2-z} dz dx \\
&= \int_{-1}^{1} \int_{0}^{1-x^2} \left(\frac{3}{2} z^2 - 6z + 6 \right) dz dx \\
&= \int_{-1}^{1} \left(\frac{1}{2} z^3 - 3z^2 + 6z \right) \Big|_{z=0}^{z=1-x^2} dx \\
&= \int_{-1}^{1} \left(\frac{1}{2}(1-x^2)^3 - 3(1-x^2)^2 + 6(1-x^2) \right) dx \\
&= \frac{184}{35}.
\end{aligned}$$

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